**Radical Equations:**

An equation involving expression under the radical sign is called a radical equation.

*e.g.*, $\sqrt{x+3} = x+1$ *and* $\sqrt{x-1} = \sqrt{x-2}+1$

# Exercise 1.4

**Solve the following equations.**

**(1)** $2x+5=\sqrt{7x+16}$

**Solution:**

$2x+5=\sqrt{7x+16} \quad ......(i)$

Squaring both sides, we get

$(2x+5)^2 = \left(\sqrt{7x+16}\right)^2$

$4x^2+20x+25=7x+16$

$4x^2+20x+25-7x-16=0$

$4x^2+20x-7x+25-16=0$

$4x^2+13x+9=0$

$4x^2+9x+4x+9=0$

$x(4x+9)+1(4x+9)=0$

$(x+1)(4x+9)=0$

*Either* $x+1=0$ *or* $4x+9=0$

$x=-1 \qquad 4x=-9$

$x=-\frac{9}{4}$

*Check* :

$$2(-1)+5=\sqrt{7(-1)+16} \qquad \Rightarrow -2+5=\sqrt{-7+16}$$

$$3=\sqrt{9} \Rightarrow 3=3 \text{ (which is true)}$$

$$Put\ x=-\frac{9}{4}\ in\ eq(i), we\ get$$

$$2\left(-\frac{9}{4}\right)+5=\sqrt{7\left(-\frac{9}{4}\right)+16}$$

$$-\frac{9}{2}+5=\sqrt{-\frac{63}{4}+16}$$

$$\frac{1}{2}=\sqrt{\frac{1}{4}}$$

$$\frac{1}{2}=\frac{1}{2}\text{(which is true)}$$

Thus, solution set = $\left\{-1,-\frac{9}{4}\right\}$

**(2)** $\sqrt{x+3}=3x-1$

**Solution:**

$$\sqrt{x+3}=3x-1 \ ......(i)$$

Squaring both sides, we get

$$\left(\sqrt{x+3}\right)^2=(3x-1)^2$$

$$x+3=9x^2-6x+1$$

$$9x^2-6x+1-x-3=0$$

$$9x^2-7x-2=0$$

$$9x^2-9x+2x-2=0$$

$$9x(x-1)+2(x-1)=0$$

$$(9x+2)(x-1)=0$$

$$Either\ 9x+2=0\ \ or\ \ x-1=0$$

$$9x=-2 \qquad x=1$$

*Check* :

*Put* $x = -\frac{2}{9}$ in eq.$(i)$, *we get*

$\sqrt{-\frac{2}{9}+3} = 3\left(-\frac{2}{9}\right)-1$

$\sqrt{\frac{25}{9}} = -\frac{2}{3}-1$

$\sqrt{\frac{25}{9}} = -\frac{5}{3}$

$\frac{5}{3} \neq -\frac{5}{3}$ (which is not true)

*Put* $x = 1$ *in* $eq(i)$, *we get*

$\sqrt{1+3} = 3(1)-1$

$\sqrt{4} = 3-1$

$2 = 2$ (which is true)

Thus, solution set = $\{1\}$

**(3)** $4x = \sqrt{13x+14}-3$

**Solution:**

$4x = \sqrt{13x+14}-3 \quad ......(i)$

$4x+3 = \sqrt{13x+14}$

Squaring both sides, we get

$(4x+3)^2 = \left(\sqrt{13x+14}\right)^2$

$16x^2+24x+9 = 13x+14$

$16x^2+24x-13x+9-14 = 0$

$16x^2+11x-5 = 0$

$16x^2+16x-5x-5 = 0$

$16x(x+1)-5(x+1) = 0$

$$\textit{Either } 16x - 5 = 0 \textit{ or } x + 1 = 0$$

$$16x = 5 \qquad x = -1$$

$$x = \frac{5}{16} \qquad x = -1$$

*Check* :

*Put* $x = \frac{5}{16}$ in eq.$(i)$, *we get*

$$4\left(\frac{5}{16}\right) = \sqrt{13\left(\frac{5}{16}\right) + 14} - 3$$

$$\frac{5}{4} = \sqrt{\frac{65}{16} + 14} - 3$$

$$\frac{5}{4} = \sqrt{\frac{289}{16}} - 3$$

$$\frac{5}{4} = \frac{17}{4} - 3$$

$$\frac{5}{4} = \frac{5}{4} \qquad (\text{which is true})$$

*Put* $x = -1$ *in eq*$(i)$, *we get*

$$4(-1) = \sqrt{13(-1) + 14} - 3$$

$$-4 = \sqrt{-13 + 14} - 3$$

$$-4 = \sqrt{1} - 3$$

$$-4 = 1 - 3$$

$$-4 \neq -2 \quad (\text{which is not true})$$

Thus, solution set = $\left\{\frac{5}{16}\right\}$

**(4)** $\sqrt{3x + 100} - x = 4$

**Solution:**

$$\sqrt{3x + 100} - x = 4 \qquad ......(i)$$

Squaring both sides, we get

$$\left(\sqrt{3x+100}\right)^2 = (x+4)^2$$

$$3x+100 = x^2+8x+16$$

$$x^2+8x+16-3x-100=0$$

$$x^2+5x-84=0$$

$$x^2+12x-7x-84=0$$

$$x(x+12)-7(x+12)=0$$

$$(x-7)(x+12)=0$$

*Either* $x-7=0$ *or* $x+12=0$

$x=7$ $\quad$ $x=-12$

*Check :*

*Put* $x=7$ in eq.$(i)$, *we get*

$\sqrt{13(7)+100}-7=4 \qquad \Rightarrow \sqrt{21+100}-7=4$

$\sqrt{121}-7=4 \qquad \Rightarrow 11-7=4$

$4=4$ (which is true)

*Put* $x=-12$ *in eq* $(i)$, *we get*

$\sqrt{3(-12)+100}-(-12)=4 \qquad \Rightarrow \sqrt{-36+100}+12=4$

$\sqrt{64}+12=4 \qquad \Rightarrow 8+12=4$

$20 \neq 4$ (which is not true)

Thus, solution set = $\{7\}$

**(5)** $\sqrt{x+5}+\sqrt{x+21}=\sqrt{x+60}$

**Solution:**

$$\left(\sqrt{x+5}+\sqrt{x+21}\right)^2 = \left(\sqrt{x+60}\right)^2$$

$$(x+5)+(x+21)+2\sqrt{(x+5)(x+21)} = x+60$$

$$x+5+x+21+2\sqrt{x^2+26x+105} = x+60$$

$$\sqrt{x+5}+\sqrt{x+21}=\sqrt{x+60} \quad ......(i)$$

Squaring both sides, we get

$$2\sqrt{x^2+26x+105}=x+60-2x-26$$
$$2\sqrt{x^2+26x+105}=-x+34$$
$$2\sqrt{x^2+26x+105}=-(x-34)$$

Squaring both sides, we get

$$\left(2\sqrt{x^2+26x+105}\right)^2=\left[-(x-34)\right]^2$$
$$4\left(x^2+26x+105\right)=x^2-68x+1156$$
$$4x^2+104x+420=x^2-68x+1156$$
$$4x^2-x^2+104x+68x+420-1156=0$$
$$3x^2+172x-736=0$$

*Here* a=3, b=172, c=-736

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-172\pm\sqrt{(172)^2-4(3)(-736)}}{2(3)}$$
$$x=\frac{-172\pm\sqrt{29584+8832}}{6}$$
$$x=\frac{-172\pm\sqrt{38416}}{6}$$
$$x=\frac{-172-196}{6} \quad or \quad x=\frac{-172+196}{6}$$
$$x=-\frac{368}{6} \qquad x=\frac{24}{6}$$
$$x=-\frac{184}{3} \qquad x=4$$

*Check* :

*Put* $x=-\frac{184}{3}$ in eq.$(i)$, *we get*

$$\sqrt{-\frac{184}{3}+5}+\sqrt{\frac{-184}{3}+21}=\sqrt{\frac{-184}{3}+60}$$

$$\sqrt{-\frac{169}{3}}+\sqrt{-\frac{121}{3}}=\sqrt{-\frac{4}{3}}$$
$$\frac{13i}{\sqrt{3}}+\frac{11i}{\sqrt{3}}=\frac{2i}{\sqrt{3}}$$
$$\frac{24i}{\sqrt{3}}=\frac{2i}{\sqrt{3}}\ (\text{which is not true})$$

*Put* $x=4$ *in eq* $(i)$, *we get*

$$\sqrt{4+5}+\sqrt{4+21}=\sqrt{4+60}$$
$$\sqrt{9}+\sqrt{25}=\sqrt{64}$$
$$3+5=8$$
$$8=8\qquad(\text{which is true})$$

Thus, solution set = $\{4\}$

**(6)** $\sqrt{x+1}+\sqrt{x-2}=\sqrt{x+6}$

**Solution:**

$$\sqrt{x+1}+\sqrt{x-2}=\sqrt{x+6}\qquad......(i)$$

Squaring both sides, we get

$$\left(\sqrt{x+1}+\sqrt{x-2}\right)^2=\left(\sqrt{x+6}\right)^2$$
$$(x+1)+(x-2)+2\sqrt{(x+1)(x-2)}=x+6$$
$$x+1+x-2+2\sqrt{x^2-x-2}=x+6$$
$$2x-1+2\sqrt{x^2-x-2}=x+6$$
$$2\sqrt{x^2-x-2}=-x+7$$
$$2\sqrt{x^2-x-2}=-(x-7)$$

Squaring both sides, we get

$$\left(2\sqrt{x^2-x-2}\right)^2=\left[-(x-7)\right]^2$$
$$4\left(x^2-x-2\right)=x^2-14x+49$$
$$4x^2-4x-8=x^2-14x+49$$

*Here* a=3, b=10, c= -57

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-10\pm\sqrt{(10)^2-4(3)(-57)}}{2(3)}$$

$$x=\frac{-10\pm\sqrt{100+684}}{6}$$

$$x=\frac{-10\pm\sqrt{784}}{6}$$

$$x=\frac{-10\pm 28}{6}$$

$$x=\frac{-10-28}{6} \quad or \qquad x=\frac{-10+28}{6}$$

$$x=\frac{-38}{6} \qquad\qquad x=\frac{18}{6}$$

$$x=-\frac{19}{3} \qquad\qquad x=3$$

*Check* :

*Put* $x=-\frac{19}{3}$ in eq. $(i)$, *we get*

$$\sqrt{-\frac{19}{3}+1}+\sqrt{\frac{-19}{3}-2}=\sqrt{-\frac{19}{3}+6}$$

$$\sqrt{-\frac{16}{3}}+\sqrt{\frac{-25}{3}}=\sqrt{-\frac{1}{3}}$$

$$\frac{4i}{\sqrt{3}}+\frac{5i}{\sqrt{3}}=\frac{i}{\sqrt{3}}$$

$$\frac{9i}{\sqrt{3}}=\frac{i}{\sqrt{3}} \qquad \text{(which is not true)}$$

*Put* $x=3$ *in eq* $(i)$, *we get*

$$\sqrt{3+1}+\sqrt{3-2}=\sqrt{3+6}$$

$$\sqrt{4}+\sqrt{1}=\sqrt{9}$$

$$2+1=3$$

$$3=3 \quad \text{(which is true)}$$

**(7)** $\sqrt{11-x}-\sqrt{6-x}=\sqrt{27-x}$

**Solution:**

$\sqrt{11-x}-\sqrt{6-x}=\sqrt{27-x} \quad ......(i)$

Squaring both sides, we get

$\left(\sqrt{11-x}-\sqrt{6-x}\right)^2=\left(\sqrt{27-x}\right)^2$

$(11-x)+(6-x)-2\sqrt{(11-x)(6-x)}=27-x$

$11-x+6-x-2\sqrt{(11-x)(6-x)}=27-x$

$17-2x-2\sqrt{x^2-17x+66}=27-x$

$-2\sqrt{x^2-17x+66}=27-x-17+2x$

$-2\sqrt{x^2-17x+66}=10+x$

Squaring both sides, we get

$\left(-2\sqrt{x^2-17x+66}\right)^2=(10+x)^2$

$4\left(x^2-17x+66\right)=100+20x+x^2$

$4x^2-68x+264=x^2+20x+100$

$4x^2-x^2-68x-20x+264-100=0$

$3x^2-88x+164=0$

*Here* a = 3, b = -88, c = 164

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{-(-88)\pm\sqrt{(-88)^2-4(3)(164)}}{2(3)}$

$x=\dfrac{88\pm\sqrt{7744-1968}}{6}$

$x=\dfrac{88\pm\sqrt{5776}}{6}$

$x=\dfrac{88\pm76}{6}$

$x = \frac{12}{6}$ $\quad$ $x = \frac{164}{6}$

$x = 2$ $\quad$ $x = \frac{82}{3}$

*Check* :

*Put* $x = 2$ in eq.$(i)$, *we get*

$\sqrt{11-2} - \sqrt{6-2} = \sqrt{27-2}$

$\sqrt{9} - \sqrt{4} = \sqrt{25}$

$3 - 2 = 5$

$1 = 5$ (which is not true)

*Put* $x = \frac{82}{3}$ *in eq*$(i)$, *we get*

$\sqrt{11 - \frac{82}{3}} - \sqrt{6 - \frac{82}{3}} = \sqrt{27 - \frac{82}{3}}$

$\sqrt{-\frac{49}{3}} - \sqrt{-\frac{64}{3}} = \sqrt{-\frac{1}{3}}$

$\frac{7i}{\sqrt{3}} - \frac{8i}{\sqrt{3}} = \frac{i}{\sqrt{3}}$

$\frac{-i}{\sqrt{3}} = \frac{i}{\sqrt{3}}$ (which is not true)

Thus, solution set = { }

**(8)** $\sqrt{4a+x} - \sqrt{a-x} = \sqrt{a}$

**Solution:**

$\sqrt{4a+x} - \sqrt{a-x} = \sqrt{a}$ ......$(i)$

Squaring both sides, we get

$\left(\sqrt{4a+x} - \sqrt{a-x}\right)^2 = \left(\sqrt{a}\right)^2$

$(4a+x) + (a-x) - 2\sqrt{(4a+x)(a-x)} = a$

$4a + x + a - x - 2\sqrt{4a^2 - 3ax - x^2} = a$

$5a - 2\sqrt{4a^2 - 3ax - x^2} = a$

$-2\sqrt{4a^2-3ax-x^2}=-4a$
$\Rightarrow \sqrt{4a^2-3ax-x^2}=2a$

Squaring both sides, we get

$\left(\sqrt{4a^2-3ax-x^2}\right)^2=(2a)^2$
$4a^2-3ax-x^2=4a^2$
$4a^2-4a^2-3ax-x^2=0$
$-3ax-x^2=0$
$-x(x+3a)=0$
$x(x+3a)=0$

*Either* $x=0$ *or* $x+3a=0$
$x=-3a$

*Check* :

*Put* $x=0$ in eq.$(i)$, *we get*
$\sqrt{4a+0}-\sqrt{a-0}=\sqrt{a}$
$\sqrt{4a}-\sqrt{a}=\sqrt{a}$
$2\sqrt{a}-\sqrt{a}=\sqrt{a}$
$\sqrt{a}=\sqrt{a}$ (which is true)

*Put* $x=-3a$ in eq.$(i)$, *we get*
$\sqrt{4a+(-3a)}-\sqrt{a-(-3a)}=\sqrt{a}$
$\sqrt{4a-3a}-\sqrt{a+3a}=\sqrt{a}$
$\sqrt{a}-\sqrt{4a}=\sqrt{a}$
$\sqrt{a}-2\sqrt{a}=\sqrt{a}$
$-\sqrt{a}=\sqrt{a}$ (which is not true)

Thus, solution set = $\{0\}$

**(9)** $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

$\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1 \quad ......(i)$

$Let\ x^2+x=y$

So eq.$(i)$ becomes

$\sqrt{y+1}-\sqrt{y-1}=1$

Squaring both sides, we get

$\left(\sqrt{y+1}-\sqrt{y-1}\right)^2=1$

$(y+1)+(y-1)-2\sqrt{(y+1)(y-1)}=1$

$y+1+y-1-2\sqrt{y^2-1}=1$

$2y-2\sqrt{y^2-1}=1$

$-2\sqrt{y^2-1}=1-2y$

Squaring both sides, we get

$\left(-2\sqrt{y^2-1}\right)^2=(1-2y)^2$

$4\left(y^2-1\right)=1-4y+4y^2$

$4y^2-4=1-4y+4y^2$

$4y^2-4-1+4y-4y^2=0$

$4y-5=0$

$y=\frac{5}{4}$

$Put\ \ y=\frac{5}{4}\ in\ x^2+x=y, we\ get$

$x^2+x=\frac{5}{4}$

$4x^2+4x=5$

$4x^2+4x-5=0$

*Here* a = 4, b = 4, c = -5

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-4\pm\sqrt{(4)^2-4(4)(-5)}}{2(4)}$$
$$x=\frac{-4\pm\sqrt{16+80}}{8}$$

$$x=\frac{-4\pm\sqrt{96}}{8}$$
$$x=\frac{-4\pm4\sqrt{6}}{8}=\frac{4\left(-1\pm\sqrt{6}\right)}{8}=\frac{-1\pm\sqrt{6}}{2}$$

Thus, solution set = $\left\{\frac{-1\pm\sqrt{6}}{2}\right\}$

**(10)** $\sqrt{x^2+3x+8}+\sqrt{x^2+3x+2}=3$

**Solution:**

$\sqrt{x^2+3x+8}+\sqrt{x^2+3x+2}=3 \qquad ......(i)$

$Let\ x^2+3x=y$

So eq.$(i)$ becomes

$\sqrt{y+8}+\sqrt{y+2}=3$

Squaring both sides, we get

$$\left(\sqrt{y+8}+\sqrt{y+2}\right)^2=(3)^2$$
$$(y+8)+(y+2)+2\sqrt{(y+8)(y+2)}=9$$
$$y+8+y+2+2\sqrt{y^2+10y+16}=9$$
$$2y+10+2\sqrt{y^2+10y+16}=9$$
$$2\sqrt{y^2+10y+16}=9-2y-10$$
$$2\sqrt{y^2+10y+16}=-2y-1$$
$$2\sqrt{y^2+10y+16}=-(2y+1)$$

$$\left(2\sqrt{y^2+10y+16}\right)^2=\left[-(2y+1)\right]^2$$
$$4\left(y^2+10y+16\right)=4y^2+4y+1$$
$$4y^2+40y+64=4y^2+4y+1$$
$$4y^2-4y^2+40y-4y+64-1=0$$
$$36y=-63$$
$$y=-\frac{63}{36}$$

*Put* $y=-\frac{63}{36}$ *in* $x^2+3x=y$, *we get*

$$x^2+3x=-\frac{63}{36}$$
$$\Rightarrow 36x^2+108x=-63$$
$$36x^2+108x+63=0$$

*Here* a=36, b=108, c=63

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-108\pm\sqrt{(108)^2-4(36)(63)}}{2(36)}$$
$$x=\frac{-108\pm\sqrt{11664+9072}}{72}$$
$$x=\frac{-108\pm\sqrt{2592}}{72}$$
$$x=\frac{-108\pm 36\sqrt{2}}{72}$$
$$x=\frac{36\left(-3\pm\sqrt{2}\right)}{72}$$
$$x=\frac{-3\pm\sqrt{2}}{2}$$

Thus, Solution set = $\left\{\frac{-3\pm\sqrt{2}}{2}\right\}$

**(11)** $\sqrt{x^2+3x+9}+\sqrt{x^2+3x+4}=5$

**Solution:**

$\sqrt{x^2+3x+9}+\sqrt{x^2+3x+4}=5 \qquad ......(i)$

$Let\ x^2+3x=y$

So eq.$(i)$ becomes

$\sqrt{y+9}+\sqrt{y+4}=5$

Squaring both sides, we get

$\left(\sqrt{y+9}+\sqrt{y+4}\right)^2=(5)^2$

$(y+9)+(y+4)+2\sqrt{(y+9)(y+4)}=25$

$y+9+y+4+2\sqrt{y^2+13y+36}=25$

$2y+13+2\sqrt{y^2+13y+36}=25$

$2\sqrt{y^2+13y+36}=25-2y-13$

$2\sqrt{y^2+13y+36}=-2y+12$

$2\sqrt{y^2+13y+36}=-2(y-6)$

$\Rightarrow \sqrt{y^2+13y+36}=-(y-6)$

Squaring both sides, we get

$\left(\sqrt{y^2+13y+36}\right)^2=\left[-(y-6)\right]^2$

$y^2+13y+36=y^2-12y+36$

$y^2-y^2+13y+12y+36-36=0$

$25y=0$

$\Rightarrow y=0$

$Put\ \ y=0\ in\ x^2+3x=y, we\ get$

$x^2+3x=y$

$x^2+3x=0$

$x(x+3)=0$

$Either\ x=0\ or\ x+3=0$

$$x = -3$$

Thus, Solution set = $\{-3, 0\}$