Quadratic Formula:

Derivation of quadratic formula by using completing square method.

The quadratic equation in standard form is

$ax^2 + bx + c = 0$, $a \neq 0$

Dividing each term of the equation by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Shifting constant term $\frac{c}{a}$ to the right, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ on both sides, we obtain

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square root on both sides, we get

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$or\ x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as "quadratic formula".

Exercise 1.2

Q1. Solve the following equations using quadratic formula:

(i) $2-x^2=7x$

Solution:

$2-x^2=7x$

$-x^2-7x+2=0$

$-\left(x^2+7x-2\right)=0$

$\Rightarrow x^2+7x-2=0$

Comparing it with standard quadratic equation, we have

$ax^2+bx+c=0$

Here a=1, b=7, c=-2

Now $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{-7\pm\sqrt{(7)^2-4(1)(-2)}}{2(1)}$

$x=\dfrac{-7\pm\sqrt{49+8}}{2}$

$x=\dfrac{-7\pm\sqrt{57}}{2}$

Thus, solution set = $\left\{\dfrac{-7\pm\sqrt{57}}{2}\right\}$

(ii) $5x^2+8x+1=0$

Solution:

$5x^2+8x+1=0$

Here $a = 5, b = 8, c = 1$

Now $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-8 \pm \sqrt{(8)^2 - 4(5)(1)}}{2(5)}$

$x = \frac{-8 \pm \sqrt{64 - 20}}{10}$

$x = \frac{-8 \pm \sqrt{44}}{10}$

$x = \frac{-8 \pm 2\sqrt{11}}{10}$

$x = \frac{2\left(-4 \pm \sqrt{11}\right)}{10}$

$x = \frac{-4 \pm \sqrt{11}}{5}$

Thus solution set = $\left\{\frac{-4 \pm \sqrt{11}}{5}\right\}$

(iii) $\sqrt{3}x^2 + x = 4\sqrt{3}$

Solution:

$\sqrt{3}x^2 + x = 4\sqrt{3}$

$\sqrt{3}x^2 + x - 4\sqrt{3} = 0$

Comparing it with standard quadratic equation, we have

$ax^2 + bx + c = 0$

Here $a = \sqrt{3}, b = 1, c = -4\sqrt{3}$

Now $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\sqrt{3}\right)\left(-4\sqrt{3}\right)}}{}$

$$x=\frac{-1\pm\sqrt{1+48}}{2\left(\sqrt{3}\right)}$$

$$x=\frac{-1\pm\sqrt{49}}{2\sqrt{3}}$$

$$x=\frac{-1\pm 7}{2\sqrt{3}}$$

$$x=\frac{-1+7}{2\sqrt{3}} \quad or \quad x=\frac{-1-7}{2\sqrt{3}}$$

$$x=\frac{6}{2\sqrt{3}} \qquad x=\frac{-8}{2\sqrt{3}}$$

$$x=\frac{3}{\sqrt{3}} \qquad x=-\frac{4}{\sqrt{3}}$$

$$x=\frac{3}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$x=\frac{3\sqrt{3}}{\left(\sqrt{3}\right)^2}$$

$$x=\frac{3\sqrt{3}}{3}$$

$$x=\sqrt{3}$$

Thus solution set = $\left\{\sqrt{3},-\frac{4}{\sqrt{3}}\right\}$

(iv) $4x^2-14=3x$

Solution:

$$4x^2-14=3x$$

$$4x^2-3x-14=0$$

Comparing it with standard quadratic equation, we have

$$ax^2+bx+c=0$$

Here $a=4, b=-3, c=-14$

Now $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-14)}}{2(4)}$$

$$x = \frac{3 \pm \sqrt{9 + 224}}{8}$$

$$x = \frac{3 \pm \sqrt{233}}{8}$$

$$x = \frac{3 \pm \sqrt{233}}{8}$$

Thus solution set = $\left\{ \frac{3 \pm \sqrt{233}}{8} \right\}$

(v) $6x^2 - 3 - 7x = 0$

Solution:

Comparing it with standard quadratic equation, we have

$ax^2 + bx + c = 0$

Here $a = 6, b = -7, c = -3$

Now $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-3)}}{2(6)}$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{12}$$

$$x = \frac{7 \pm \sqrt{121}}{12}$$

$$x = \frac{7 \pm 11}{12}$$

$$x = \frac{7 + 11}{12} \quad or \quad x = \frac{7 - 11}{12}$$

$$x = \frac{18}{12} \qquad x = -\frac{4}{12}$$

Thus solution set = $\left\{-\frac{1}{3}, \frac{3}{2}\right\}$

(vi) $3x^2 + 8x + 2 = 0$

Solution:

$3x^2 + 8x + 2 = 0$

Comparing it with standard quadratic equation, we have

$ax^2 + bx + c = 0$

Here $a = 3, b = 8, c = 2$

Now $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-8 \pm \sqrt{(8)^2 - 4(3)(2)}}{2(3)}$

$x = \dfrac{-8 \pm \sqrt{64 - 24}}{6}$

$x = \dfrac{-8 \pm \sqrt{40}}{6}$

$x = \dfrac{-8 \pm 2\sqrt{10}}{6}$

$x = \dfrac{2\left(-4 \pm \sqrt{10}\right)}{6}$

$x = \dfrac{-4 \pm \sqrt{10}}{3}$

Thus solution set = $\left\{\dfrac{-4 \pm \sqrt{10}}{3}\right\}$

(vii) $\dfrac{3}{x-6} - \dfrac{4}{x-5} = 1$

Solution:

$$\frac{3(x-5)-4(x-6)}{(x-6)(x-5)}=1$$

$$3x-15-4x+24=(x-6)(x-5)$$

$$-x+9=x^2-11x+30$$

$$x^2-11x+x+30-9=0$$

$$x^2-10x+21=0$$

Comparing it with standard quadratic equation, we have

$ax^2+bx+c=0$

Here $a=1, b=-10, c=21$

Now $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(21)}}{2(1)}$$

$$x=\frac{10\pm\sqrt{100-84}}{2}$$

$$x=\frac{10\pm\sqrt{16}}{2}$$

$$x=\frac{10\pm4}{2}$$

$$x=\frac{10+4}{2} \quad or \quad x=\frac{10-4}{2}$$

$$x=\frac{14}{2} \qquad x=\frac{6}{2}$$

$$x=7 \qquad x=3$$

Thus, solution set = {3,7}

(viii) $\dfrac{x+2}{x-1}-\dfrac{4-x}{2x}=2\dfrac{1}{3}$

Solution:

$$\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}$$

$$\frac{(2x^2+4x)-(4x-x^2-4+x)}{2x^2-2x}=\frac{7}{3}$$

$$\frac{(2x^2+4x)-(-x^2+5x-4)}{2x^2-2x}=\frac{7}{3}$$

$$\frac{2x^2+4x+x^2-5x+4}{2x^2-2x}=\frac{7}{3}$$

$$7(2x^2-2x)=3(3x^2-x+4)$$

$$14x^2-14x=9x^2-3x+12$$

$$14x^2-9x^2-14x+3x-12=0$$

$$5x^2-11x-12=0$$

Comparing it with standard quadratic equation, we have

$$ax^2+bx+c=0$$

$Here\ \ a=5, b=-11, c=-12$

Now $\quad x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-(-11)\pm\sqrt{(-11)^2-4(5)(-12)}}{2(5)}$$

$$x=\frac{11\pm\sqrt{121+240}}{10}$$

$$x=\frac{11\pm\sqrt{361}}{10}$$

$$x=\frac{11\pm 19}{10}$$

$$x=\frac{11+19}{10} \quad or \quad x=\frac{11-19}{10}$$

$$x=\frac{30}{10} \qquad x=-\frac{8}{10}$$

$$x=3 \qquad x=-\frac{4}{5}$$

Thus, solution set = $\left\{3,-\frac{4}{5}\right\}$

(ix) $\frac{a}{x-b}+\frac{b}{x-a}=2$

Solution:

$$\frac{a}{x-b}+\frac{b}{x-a}=2$$

$$\frac{a(x-a)+b(x-b)}{(x-a)(x-b)}=2$$

$$ax-a^2+bx-b^2=2(x-a)(x-b)$$

$$ax+bx-a^2-b^2=2(x^2-ax-bx+ab)$$

$$ax+bx-a^2-b^2=2x^2-2ax-2bx+2ab$$

$$2x^2-2ax-ax-2bx-bx+2ab+a^2+b^2=0$$

$$2x^2-3ax-3bx+2ab+a^2+b^2=0$$

$$2x^2-3(a+b)x+(2ab+a^2+b^2)=0$$

$$2x^2-3(a+b)x+(a+b)^2=0$$

Comparing it with standard quadratic equation, we have

$ax^2+bx+c=0$

Here $a=2, b=-3(a+b), c=(a+b)^2$

Now $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-[-3(a+b)]\pm\sqrt{[-3(a+b)]^2-4(2)(a+b)^2}}{2(2)}$$

$$x=\frac{3(a+b)\pm\sqrt{9(a+b)^2-8(a+b)^2}}{4}$$

$$x=\frac{3(a+b)\pm\sqrt{(a+b)^2}}{4}$$

$$x=\frac{3(a+b)\pm(a+b)}{4}$$

$$x=\frac{3(a+b)+(a+b)}{4} \qquad x=\frac{3(a+b)-(a+b)}{4}$$

$$x=\frac{3a+3b+a+b}{} \qquad x=\frac{3a+3b-a-b}{}$$

$$x = \frac{4a+4b}{4} \qquad x = \frac{2a+2b}{4}$$

$$x = \frac{4(a+b)}{4} \qquad x = \frac{2(a+b)}{4}$$

$$x = a+b, \qquad x = \frac{1}{2}(a+b)$$

Thus, solution set = $\left\{(a+b), \frac{1}{2}(a+b)\right\}$

(x) $-(l+m)-lx^2+(2l+m)x=0\ ,\ l \neq 0$

Solution:

$$-(l+m)-lx^2+(2l+m)x=0$$
$$-lx^2+(2l+m)x-(l+m)=0$$
$$-\left[lx^2-(2l+m)x+(l+m)\right]=0$$
$$lx^2-(2l+m)x+(l+m)=0$$

Comparing it with standard quadratic equation, we have

$ax^2+bx+c=0$

Here $a=l, b=-(2l+m), c=(l+m)$

Now $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x = \frac{-\left[-(2l+m)\right] \pm \sqrt{\left[-(2l+m)\right]^2-4(l)(l+m)}}{2l}$$

$$x = \frac{(2l+m) \pm \sqrt{(2l+m)^2-4l(l+m)}}{2l}$$

$$x = \frac{(2l+m) \pm \sqrt{4l^2+4lm+m^2-4l^2-4lm}}{2l}$$

$$x = \frac{(2l+m) \pm \sqrt{m^2}}{2l}$$

$$x = \frac{(2l+m) \pm m}{2l}$$

$x = \frac{2l + 2m}{2l}$, $\quad x = \frac{2l + m - m}{2l}$

$x = \frac{2(l + m)}{2l}$, $\quad x = \frac{2l}{2l}$

$x = \frac{l + m}{l}$, $\quad x = l$

Thus, solution set = $\left\{ \frac{l + m}{l}, l \right\}$